Prudent Equilibria and Strategic Uncertainty in Discontinuous Games

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Abstract

We introduce the new concept of prudent equilibrium to model strategic uncertainty, and prove it exists in large classes of discontinuous games. When the game is better-reply secure, we show that prudent equilibrium refines Nash equilibrium. In contrast with the current literature, we don't use probabilities to model players' strategies and beliefs about other players' strategies. We provide examples (first-price auctions, location game, Nash demand game, etc.) where the prudent equilibrium is the intuitive solution of the game.

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1 Introduction

Consider a first-price sealed-bid auction with complete information between two bidders. The players are characterized by their valuation v_1 and v_2 of the item for sale, $v_1 < v_2$, and they are supposed to choose bids x_1 and x_2 in $[0, M]$, $M > 0$. Assume that in case of ties, i.e. if $x_1 = x_2$, then the winner is the player with the highest value. An easy computation proves that for every $x \in [v_1, v_2]$, the strategy profile (x, x) is a Nash equilibrium of this strategic game. Yet, for $x > v_1$, these equilibria represent fragile situations, because of *strategic uncertainty*: if player 2 does not respect his equilibrium strategy and decreases slightly his bid, then player 1 gets the item for a price x higher than his valuation v_1. As a matter of fact, any strategy $x_1 \leq v_1$ is also a best-reply of player 1 if player 2 plays $x > v_1$, but x_1 is immune to a small modification of player 2's strategy. In addition, if player 2 is supposed to play $x > v_1$, and if he predicts that his opponent should play $x_1 \leq v_1$, then he could be tempted to lower his bid x in order to increase his payoff. Thus, playing $x > v_1$ for player 1 seems definitely a bad choice, even if the other player is assumed to play the same strategy.

This example illustrates that Nash equilibrium concept has to be refined in order to keep some predictive power. Note that the existence of a pure Nash equilibrium in such game can be obtained from

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Reny's theorem [23], which guarantees existence for the large class of better-reply secure games. This class encompasses many discontinuous economic games, in particular the first-price auction described above. Astonishingly, there is no refinement notion of pure strategy Nash equilibrium for discontinuous games which covers this example, or more generally the class of better-reply secure games. A possible reason is that most refinement notions - like perfect equilibrium of Selten [27] - require the existence of equilibria of some auxiliary games, where the players' strategies are perturbed by random mistakes, and payoffs are expected payoffs. If the initial game is better-reply secure but discontinuous, such auxiliary games are in general neither better-reply secure nor quasiconcave, thus general existence result in pure strategies cannot be used.[1]

In this paper, we introduce a new refinement of pure strategy Nash equilibrium in discontinuous games, called prudent equilibrium. We prove its existence (Theorem 15) for the class of *p-robust games*, which contains many discontinuous economic games. Roughly, a game is p-robust if for every joint strategy x, no player can increase largely his payoff at x by arbitrary small modifications of his strategy, even if the other players can change slightly their strategy. The first-price auction above is p-robust: for example, at every strategy profile (x, x) with $x < v_1$, if player 1 increases his strategy a little bit, player 2 can answer by the same modification, so that player 1 does not increase his payoff.

We now provide an informal definition of our main solution concept, *prudent equilibrium.* The main issue in the introductory example is strategic uncertainty, i.e. the uncertainty related to other players' strategies and rationality (see Brandenburger [5]). A radical way to remove strategic uncertainty in games would be to consider extremely prudent players, who try to maximize $\tilde{u}_i(x) = \inf_{x_{-i} \in X_{-i}} u_i(x_i, x_{-i})$ with respect to their strategy x_i, u_i being the initial payoff function of player i, and X_{-i} the strategy sets of his opponents. A less extreme answer would be to assume that given player i's belief about the potential strategy profile x_{-i} of his opponents, he has good reasons to think that their true strategies will stay in some set $Y_{-i}(x_{-i}) \subset X_{-i}$. Then, a prudent player would choose his strategy x_i in order to maximize $\tilde{u}_i(x) = \inf_{y_{-i} \in Y_{-i}(x_{-i})} u_i(x_i, y_{-i})$. This function can also be written as

$$\tilde{u}_i(x) = \inf_{y_{-i} \in X_{-i}} (u_i(x_i, y_{-i}) + \delta(y_{-i}, x_{-i}))$$

where

$$\delta(x_{-i}, y_{-i}) = \begin{cases} 0 & \text{if } y_{-i} \in Y_{-i}(x_{-i}), \\ +\infty & \text{otherwise.} \end{cases}$$

In the infimum above, note that only the strategies y_{-i} for which $\delta(y_{-i}, x_{-i})$ is equal to 0 are useful, and the other one, for which $\delta(y_{-i}, x_{-i})$ is infinite, can be removed. In short, δ is a measure for player i of the importance of a potential deviation y_{-i} from the the expected strategy profile x_{-i}.

A natural generalization leads to the definition of the following auxiliary "prudent" game

$$u_i^{\lambda}(x) = \inf_{y_{-i} \in X_{-i}} (u_i(x_i, y_{-i}) + \frac{c_i(y_{-i}, x_{-i})}{\lambda}) \tag{1}$$

where $c_i : X_{-i} \times X_{-i} \to \mathbb{R}$, and $\lambda > 0$ is some normalization coefficient. The function $\frac{c_i(x_{-i}, y_{-i})}{\lambda}$ could be interpreted as a (free) insurance payed to player i if the other players $-i$ play y_{-i} instead

[1] Carbonnel ([7], [6]) extends the existence of perfect equilibria in mixed strategies for some particular classes of discontinuous games. Andersson et al. [2] have recently introduced a pure strategy refinement of Nash equilibrium which has cutting power in some discontinuous games, but whose existence is guaranteed only in continuous games.

of the expected strategy x_{-i}, and $1/\lambda$ would parametrize the insurance level. By analogy with the previous case, the function $\frac{c_i(y_{-i},x_{-i})}{\lambda}$ is also a way to parametrize strategic uncertainty: $x_{-i} \in X_{-i}$ is the strategy expected by player i, and $\frac{c_i(y_{-i},x_{-i})}{\lambda} \leq \frac{c_i(y'_{-i},x_{-i})}{\lambda}$ means that the possible deviation y_{-i} has more importance for player i than y'_{-i} has.

Interestingly, the prudent behaviour described above has a smoothing effect on the initial game: in general, the prudent payoff functions $\tilde{u}_i$ are more regular than the initial payoff functions u_i. This smoothing effect implies that for every p-robust game G, and for a large class of functions c_i, there exists a Nash equilibrium of the prudent game associated to G (Theorem 11). This opens a route for refinement, and indeed, we prove that if the initial game G is better-reply secure, and if the level of insurance $1/\lambda$ tends to $+\infty$, then any limit point of Nash equilibria of the prudent games is a Nash equilibrium of G (Proposition 14). We call this limit a prudent equilibrium, and we prove it is a refinement of Nash equilibrium. For example, in the first-price auction above, the only prudent equilibrium is the intuitive solution (v_1, v_1).

Our definition of a prudent game should be compared to variational preferences, introduced by Maccheroni, Marinacci and Rustichini [17] to model uncertainty aversion in decision theory. Recall that variational preferences on the set of acts $\mathcal{F}$ are represented by

$$V(f) = \min_{p \in \Delta}\left(\int u(f)dp + c(p)\right),$$

where u is a utility function, $f \in \mathcal{F}$ an act, Δ the set of priors over a state space S, and $c : \Delta \to [0, +\infty]$ an index of uncertainty aversion. The interpretation by Maccheroni et al. is the following. When the decision maker contemplates choosing an act f, the malevolent Nature tries to minimize its expected utility. Any prior p can be chosen, but Nature must pay a cost $c(p)$ to do so. In their setting, this cost is also interpreted as an ambiguity index.

Our model adapts[2] some of these ideas to a strategic setting. In particular, ambiguity is turned into strategic uncertainty. But a major difference is that variational preferences are valid in a probabilistic setting, although we consider only a deterministic framework. Also, the cost $c_i(x_{-i}, y_{-i})$ in our model depends on potential strategy profile y_{-i} of $-i$, which plays the role of p in variational preferences, but also on the strategy profile x_{-i} expected by i.

Many other papers have tried to model strategic uncertainty in games. In quantal-response equilibrium models, pioneered by Kelvey and Palfrey [19], strategic uncertainty is represented by a probability distribution (some noise) added to the initial payoff of each player, which defines a perturbed game. For every mixed strategy profile σ, every player i acts optimally in the perturbed game against σ_{-i}. This induces another probability distribution over the observed actions of the players. If this probability distribution is σ, it is, by definition, a quantal-response equilibrium. In a similar vein, Andersson et al. [2] consider that players choose pure strategies, and strategic uncertainty is now represented through probabilistic subjective beliefs about the strategies of each player's opponents. As above, this defines an equilibrium notion in some auxiliary noisy game. Remark that both approaches are related to refinement literature (see Selten [27] or Myerson [21]), and as a matter of fact, when the level of noise tends to zero, they provide refinement concepts.

[2] De marco and Romaniello [10] have adapted Maccheroni, Marinacci and Rustichini [17] to a Bayesian strategic setting. Their model and their scope is not connected with ours.

Non additive models are an alternative to model strategic uncertainty. If strategic uncertainty is represented by a set of priors, then the preferences of each players can be defined through Choquet expected utility model (see Mukerji [20], Marinacci [18] , Ryan [26], or Eichberger and Kelsey [12] who also model optimism or pessimism in strategic games), or through Gilboa-Schmeidler maximin model (Klibanov [14], Dow-Verlang [11], Lo [16] or De Marco and Romaniello [10]). Most papers above differ in their definition of the support for the beliefs. Recently, Renou and Schlag [22] have proposed a dual model based on minimax behaviour: in their approach, regret guides players in forming probabilistic assessments and, ultimately, in making choices.

The main difference between our model and these models is that beliefs about the strategies of the other players are not represented by sets of priors, but by deterministic functions. It turns out to be a very tractable approach in many cases, even when the initial game is discontinuous (see Section 5), and it has several interpretations. For readers who are more interested in mixed strategies, note that our model can also be applied to the mixed extension of a game. We give only one example to illustrate this point (Example 35), and more generally, prudent equilibria in mixed strategies will be studied in a separate paper.

The paper is organized as follows. Section 2 introduces p-robustness and prudent games. A measure of strategic uncertainty is defined, together with some local comparison index of strategic uncertainty between players. In addition, we characterize the class of prudent payoffs. Section 3 introduces prudent equilibrium, and proves it is a refinement of Nash equilibrium in better-reply secure games. Section 4 proposes some extensions, for example when the game possesses enough symmetry. Section 5 provides examples. Mathematical proofs are given in the last section.

2 The main solution concepts

2.1 The general framework

There are N players.[3] The pure strategy set of each player $i \in N$, denoted by X_i, is a non-empty, compact subset of a metric topological vector space (E_i, d_i). Each player i has a bounded payoff function

$$u_i : X = \prod_{i \in N} X_i \to \mathbb{R}.$$

A strategic game G is a pair $G = ((X_i)_{i\in N}, (u_i)_{i\in N})$. For every $x \in X$ and every $i \in N$, we denote $x_{-i} = (x_j)_{j\neq i}$ and $X_{-i} = \Pi_{j\neq i} X_j$. Throughout this paper, a game G satisfying the above assumptions is called a compact game. Additionally, G is called a quasiconcave game if for every player i, X_i is convex, and if for every player i and every strategy $x_{-i} \in X_{-i}$, $u_i(x_i, x_{-i})$ is quasiconcave in x_i. The game G is called continuous if for every player i, u_i is continuous in x.

We shall denote by $\Gamma = \overline{\{(x, u(x)) : x \in X\}}$ the closure of the graph of G. Let us define the "secure payoff level" of player i when he plays d_i and when the other players play x_{-i} by $\underline{u_i}(d_i, x_{-i}) = \liminf_{x'_{-i}\to x_{-i}} u_i(d_i, x'_{-i})$. Following Reny [23], the game G is *better-reply secure* if whenever $(x, v) \in \Gamma$ and x is not a Nash equilibrium, some player $i \in N$ can secure a payoff strictly above v_i, i.e. there exists $d_i \in X_i$ such that $\underline{u_i}(d_i, x_{-i}) > v_i$. It is easy to check that every continuous game is better-reply secure,

[3] According to the context, N will denote the set of players or its cardinal.

and we recall that every better-reply quasiconcave game admits a Nash equilibrium (Reny's theorem [23]).

2.2 P-robustness

The following definition plays a central role in all our results.

Definition 1. *A payoff function u_i is p-robust at $x \in X$ if for every $\varepsilon > 0$ and for every neighborhood $V_{x_{-i}}$ of x_{-i}, there exists some open neighborhood V_{x_i} of x_i such that* $\sup_{x'_i \in V_{x_i}} \inf_{x'_{-i} \in V_{x_{-i}}} u(x'_i, x'_{-i}) \leq u_i(x) + \varepsilon$. *The payoff function u_i is p-robust if this holds for every $x \in X$. If for every $i \in N$, u_i is p-robust, then we say that G is p-robust.*

Thus, u_i is p-robust if player i cannot largely improve his payoff by modifying slightly his strategy, if he anticipates the worst local possible modification of other players' strategies. If u_i was not p-robust at x, a pessimistic player could have some incentive to slightly modify his strategy x_i. In particular, games with continuous payoff functions are p-robust. Many discontinuous economic games are p-robust. As remarked in the introduction, first-price auctions are p-robust, which remains true in case of equal sharing rule (although in this case this game is no more better-reply secure). Other examples are second-price auctions, Bertrand's price competition, Cournot's model of oligopoly, Hotelling's model of spatial competition, timing game, etc.

Example 2. Consider a game $G = ((X_i)_{i\in N}, (u_i)_{i\in N})$, let $x \in X$, and assume that for every neighborhood $V_{x_{-i}}$ of x_{-i}, there exists some mapping ψ_{-i} from some neighborhood of x_i into $V_{x_{-i}}$, continuous at x_i, with $\psi_{-i}(x_i) = x_{-i}$, and such that $x'_i \to u_i(x'_i, \psi_{-i}(x'_i))$ is upper semicontinuous on a neighborhood of x_i. Then u_i is p-robust at x. Indeed, upper semicontinuity assumption implies that for every $\varepsilon > 0$, there exists some open neighborhood V_{x_i} of x_i such that $\sup_{x'_i \in V_{x_i}} u_i(x'_i, \psi_{-i}(x'_i)) \leq u_i(x) + \varepsilon$, which implies p-robustness (from ψ_{-i} continuous at x_i, taking V_{x_i} smaller if necessary). Considering the particular case where ψ_{-i} is the constant mapping equal to x_{-i}, we get that if u_i is upper semicontinuous in x_i, then it is p-robust. The criterium above can be applied, for example, to the first-price auctions described in the introduction: at every $(x, x) \in [0, 1] \times [0, 1]$, we can choose $\psi_{-i}(x'_i) = x'_i$.

P-robustness assumption is fundamentally different from better-reply security or its generalizations. As the following example shows, there is no direct relationship between better-reply security and p-robustness.

Example 3. Consider a two-player game with $X_1 = X_2 = [0, 1]$, $u_1(x_1, x_2) = -(x_1 - x_2)^2$ for every $(x_1, x_2) \in [0, 1]^2$, $u_2(0, x_2) = x_2$ for every $x_2 \in [0, 1]$ and $u_2(x_1, x_2) = -x_2$ for every $(x_1, x_2) \in]0, 1] \times [0, 1]$. This game is p-robust, but it is not better-reply secure (because it is quasiconcave and does not possess any Nash equilibria). Conversely, define $v_1(0, 0) = 0$, $v_1(x_1, x_2) = 1$ for every $(x_1, x_2) \neq (0, 0)$, and $v_2(x_1, x_2) = x_2$. This game is better-reply secure, but v_1 is not p-robust at $(0, 0)$.

Example 4. A game $G = ((X_i)_{i\in N}, (u_i)_{i\in N})$ is called piecewise upper semicontinuous if for every player i, there exist a partition $(X^{k,i})_{k\in K_i}$ of X such that for every $k \in K_i$:

(1) The multivalued function $x_i \to \{x_{-i} \in X_{-i} : (x_i, x_{-i}) \in X^{k,i}\}$ is lower semicontinuous.[4]

[4]Let A and B be two topological spaces. A multivalued function Φ from A to B is lower semicontinuous if for every open subset V of B, the set $\{x \in A : \Phi(x) \cap V \neq \emptyset\}$ is an open subset of A.

(2) The restriction of u_i to $X^{k,i}$ is upper semicontinuous.

In Appendix 6.1, it is proved that a piecewise upper semicontinuous game is p-robust. Many discontinuous games are piecewise continuous, and the following criterium is, in general, straighforward to check in such cases. In particular, it can be simpler to verify than the criterium of Example 2, which requires to be able to define the auxiliary mapping ψ_i.

Example 5. Consider the following diagonal game (see Bich and Laraki [4]): for every $i \in N$, we let f_i, g_i be *upper semicontinuous* mappings from $[0,1] \times [0,1]$ to $\mathbb{R}$, and $h_i : [0,1]^N \to \mathbb{R}$. The payoff of player i is:

$$u_i(x_i, x_{-i}) = \begin{cases} f_i(x_i, \phi(x_{-i})) & \text{if } \phi(x_{-i}) > x_i, \\ g_i(x_i, \phi(x_{-i})) & \text{if } \phi(x_{-i}) < x_i, \\ h_i(x_i, x_{-i}) & \text{if } \phi(x_{-i}) = x_i, \end{cases}$$

where $\phi : [0,1]^{N-1} \to [0,1]$ is a continuous mapping such that its (multivalued) inverse $\phi^{-1}(x_i) = \{x_{-i} \in X_{-i} : x_i = \phi(x_{-i})\}$ is lower semicontinuous. This assumption is satisfied, for example, when ϕ is one of the following mapping: $\phi_1(y) = \max\{y_1, y_2, ..., y_{N-1}\}$, $\phi_2(y) = \min\{y_1, y_2, ..., y_{N-1}\}$, $\phi_3(y) = \frac{1}{N-1}\sum_{j=1}^{N-1} y_j$, or $\psi_k(y) = \{k\text{-th highest value of } \{y_1, ..., y_{N-1}\}\}$, $k = 1, ..., N-1$. Such functions encompasse many models of competition with complete information (e.g., some auctions for $\phi = \phi_1$, wars of attrition, preemption or Bertrand competition for $\phi = \phi_2$).

If h_i is upper semicontinuous for every $i \in N$, then the diagonal game is piecewise continuous, thus it is p-robust. Indeed, for every i, we can define $X^{i,1} = \{x \in X : x_i > \phi(x_{-i})\}\}$, $X^{i,2} = \{x \in X : x_i < \phi(x_{-i})\}\}$ and $X^{i,3} = \{x \in X : x_i = \phi(x_{-i})\}$. Then, the multivalued functions $x_i \to \{x_{-i} \in X_{-i} : x_i > \phi(x_{-i})\}$ and $x_i \to \{x_{-i} \in X_{-i} : x_i < \phi(x_{-i})\}$ have open graphs, thus are lower semicontinuous. Moreover, by assumption, the multivalued function $x_i \to \{x_{-i} \in X_{-i} : x_i = \phi(x_{-i})\}$ is lower semicontinuous, which finally proves piecewise continuity.

Another example of interest is the following case[5]: assume there are two players, $\phi_1 = \phi_2$ is equal to identity, and for every $x \in [0,1]$, $h_i(x,x) \geq \min\{f_i(x,x), g_i(x,x)\}$. Under these assumptions, we get a p-robust game. Indeed, consider for example the case $(x,x) \in]0,1[\times]0,1[$ and $h_i(x,x) \geq f_i(x,x)$. Define $\psi_{-i}(x'_i) = 2x'_i - x$ if $x'_i > x$ and $\psi_{-i}(x'_i) = x$ if $x'_i \leq x$. Then the mapping $x'_i \to u_i(x'_i, \psi_{-i}(x'_i))$ is upper semicontinuous on a neighborhood of x, thus we can apply the criterium of Example 2. The other cases are similar.

2.3 Solution concept

2.3.1 $\lambda-$equilibrium

To every game we can associate an auxiliary "prudent" game as follows:

Definition 6. *Let $\mathcal{F}_i$ be the set of continuous[6] real-valued functions c from $X_{-i} \times X_{-i}$ to $[0, +\infty]$ such that $c(x_{-i}, y_{-i}) = 0$ if and only if $x_{-i} = y_{-i}$. Let $c = (c_i)_{i \in N} \in \Pi_{i \in N}\mathcal{F}_i$. For every $\lambda > 0$, the $\lambda-$prudent game associated to $G = ((X_i)_{i \in N}, (u_i)_{i \in N})$ is $G^\lambda = ((X_i)_{i \in N}, (u_i^\lambda)_{i \in N})$, where for each player $i \in N$,*

$$u_i^\lambda(x_i, x_{-i}) = \inf_{y_{-i} \in X_{-i}} \left\{ u_i(x_i, y_{-i}) + \frac{c_i(x_{-i}, y_{-i})}{\lambda} \right\}. \tag{2}$$

[5] It can be applied, for example, to non-zero-sum, noisy games of timing (see Example 3.1. in [23]).

[6] The set $[0, +\infty]$ is endowed with the topology induced by the usual topology of the extended real line $[-\infty, +\infty]$.

A $\lambda-$prudent-equilibrium (or $\lambda-$equilibrium[7]) of G is a Nash equilibrium of G^{λ}.

The first important properties of the auxiliary game G^{λ} are given in the following proposition. The proof can be found in Appendix 6.2.

Proposition 7. For every quasiconcave game G:

1. G^{λ} is quasiconcave.
2. For every $x_i \in X_i$, $x_{-i} \to u_i^{\lambda}(x_i, x_{-i})$ is continuous.
3. If G is p-robust, then u_i^{λ} is upper semicontinuous with respect to x.
4. $u_i^{\lambda} \leq \underline{u}_i \leq u_i$.

For example, a natural choice in Definition 6 is $c_i(x_{-i}, y_{-i}) = d_{-i}(x_{-i}, y_{-i})^{\alpha_i}$ for some $\alpha_i > 0$, where d_{-i} is any product distance on X_{-i} of the d_j $(j \neq i)$. The infimum in the definition of u_i^{λ} means that each player of G^{λ} is prudent (or pessimistic) with respect to the rationality[8] of opponents. The function $\frac{c_i(x_{-i},y_{-i})}{\lambda}$ allows to weight differently opponents' actions, and could be interpreted in several ways: as discussed in the introduction, it may be seen as a functional index related to strategic uncertainty of player i about other players' strategies. Another interpretation is that it is a (free) insurance payed to player i if the other players $-i$ play y_{-i} instead of the predicted strategy x_{-i}. This "insurance" can be seen as an abstract way to modelize the degree of confidence[9] that player i has in his belief that opponents will respect their Nash equilibrium strategies. A last interpretation is related to ambiguity: the set of strategies of opponents can be seen as a set of deterministic priors, and $\frac{c_i(x_i,y_{-i})}{\lambda}$ is a measure of ambiguity on the other players' strategies. In particular, $\frac{c_i(x_i,y_{-i})}{\lambda} = 0$ means a maximal ambiguity (which implies that player i will act as a maximin player), and $\frac{c_i(x_i,y_{-i})}{\lambda} = +\infty$ that there is no ambiguity at all.

In each interpretation above, the functions c_i parametrize some local shape (of degree of confidence, ambiguity index, etc.), and $\lambda > 0$ parametrizes the level of the ambiguity. The case $\lambda \to 0$ corresponds to perfect insurance against strategic uncertainty, or perfect confidence, or minimal ambiguity. On the opposite, when $\lambda \to +\infty$, players are getting closer to maximin agents, which corresponds to maximal ambiguity or minimal confidence level.

This could be formalized as follows:[10] first recall that a maximin strategy for player $i \in N$ is a strategy $x_i^* \in X_i$ such that

$$\inf_{x_{-i} \in X_{-i}} u_i(x_i^*, x_{-i}) = \sup_{x_i \in X_i} \inf_{x_{-i} \in X_{-i}} u_i(x_i, x_{-i})$$

A maximin equilibrium of G is a profile of strategies $x = (x_i)_{i \in N}$ such that x_i is a maximin strategy for every player i.

[7] The notion of $\lambda-$equilibrium is actually parametrized by λ and c, but in many applications, we will be interested by comparative static effects of λ for a fixed c.

[8] We quote Aumann and Brandenburger [3]: "Suppose that each player is rational (i.e. he maximizes his utility given his beliefs), knows his own payoff function, and knows the strategy choices of the others. Then the players' choices constitute a Nash equilibrium in the game being played." This explains what is meant by a rational player or a rational action in this paper. Naturally, such definition of rationality might be discussed: see, for example, [25]

[9] In [12], Eichberger and Kelsey use capacities to modelize ambiguity and degree of confidence in a strategic game.

[10] A similar idea that maximin behavior in a game can be supported by beliefs that show extreme ambiguity has first been proposed in [18].

Proposition 8. 1. For every p-robust game[11] G, any limit of λ^n-equilibria where λ^n tends to $+\infty$ is a maximin equilibrium.

2. For every better-reply secure game G, the limit of λ^n-equilibria where λ^n tends to 0 is a Nash equilibrium.

The proof can be found in Appendix 6.3

If $X_i = X_j$ for every $(i, j) \in N^2$, then we can introduce the following index which compares locally the prudent behaviour of two players. Applications can be found in Example 23 or in Example 27.

Definition 9. *Suppose that $X_i = X_j$ for every $(i, j) \in N^2$. We define the relative prudence $p_{i|j}(x)$ of player i with respect to player j at $x \in X_1^{N-1}$, when it exists, by*

$$p_{i|j}(x) = \lim_{(x',x'')\to(x,x)} \frac{c_j(x', x'')}{c_i(x', x'')}.$$

If this limit is equal to 0, we say that Player j is infinitely more prudent than player i at $x \in X_1^{N-1}$.

Given a strategy profile of opponents x, if $p_{i|j}(x)$ increases, then prudence of player i increases (compared with that of player j) *locally around* x. Prudence measures how much each player takes into account possible modifications of other players' strategies around x. For example, if X_i is a normed space for every $i \in N$, $c_j(x', x'') = \|x' - x''\|^k$ and $c_i(x', x'') = \|x' - x''\|^l$ with $k > l$, then player j is infinitely more prudent than player i.

2.4 Characterizing our class of utility functions

Given some function $c_i \in \mathcal{F}_i$, the following proposition characterizes payoff functions v_i that are "prudent payoffs", i.e. which can be written

$$v_i(x_i, x_{-i}) = \inf_{y_{-i}\in X_{-i}} \left\{ u_i(x_i, y_{-i}) + \frac{c_i(y_{-i}, x_{-i})}{\lambda} \right\} \tag{3}$$

for some other payoff function u_i.

Proposition 10. Assume that $c_i \in \mathcal{F}_i$ defines a distance on $X_{-i} \times X_{-i}$. Let v_i be a quasiconcave payoff function. There exists a quasiconcave payoff function u_i satisfying Equation 3 if and only if

$$\forall (x, y_{-i}) \in X \times X_{-i}, \ | v_i(x_i, x_{-i}) - v_i(x_i, y_{-i}) | \leq \frac{c_i(y_{-i}, x_{-i})}{\lambda}. \tag{4}$$

The proof can be found in Appendix 6.4.

2.5 Existence of $\lambda-$equilibria in piecewise u.s.c. games

We now state a first important result of this paper.

[11] If we remove p-robustness, we can get similarly that any limit of λ^n-equilibria where λ^n tends to $+\infty$ is a sequence of ε^n-maximin equilibra for some sequence ε^n converging to 0.

Theorem 11. *Let $c = (c_i)_{i\in N} \in \Pi_{i\in N}\mathcal{F}_i$ and G be a quasiconcave and p-robust game. For every $\lambda > 0$, there exists a $\lambda-$equilibrium.*

Indeed, from Proposition 7, G^λ is a compact and quasiconcave game, u_i^λ is upper semicontinuous with respect to x and continuous with respect to x_{-i}. Thus, for every $\lambda > 0$, G^λ possesses a Nash equilibrium (see, for example, Theorem 2 in [8]) .

3 Prudent equilibrium and refinement of Nash equilibrium in better-reply secure games

Refinement theory refers to the selection of particular equilibria which are more plausible. Many refinement concepts exist in literature: in his seminal paper, Selten [27] introduces trembling hand perfect equilibrium and proves its existence in finite-strategy games. Roughly, the idea is to select equilibria which are immune to small mistakes of the other players. Here, a mistake is formalized by a mixed strategy close to the initial strategy. Perfect equilibria have been refined in several direction: Myerson [21] introduces proper equilibria, where players are more likely to make mistake in direction that are least harmfull to them. Kohlberg and Mertens [15] defines stable-equilibrium notion, which requires stronger requirements than perfect equilibrium; Simon and Stinchcombe [28] extend perfect and proper equilibria to infinite normal-form games. [12]

To the best of our knowledge, there is no general existence result of refined Nash equilibrium in discontinous games with pure strategies. This section proposed such a result.[13]

Definition 12. *A strategy profile $x \in X$ is a prudent equilibrium of G if there exists $c = (c_i)_{i\in N} \in \Pi_{i\in N}\mathcal{F}_i$ such that x is the limit of λ^n-equilibria for $\lambda^n \to 0$. The strategy profile x is a strictly prudent equilibrium if this holds for any $c \in \mathcal{F}$.*

Remark 13. Andersson et al. [2] were the first to propose a notion of robustness to strategic uncertainty. They consider a family $\mathcal{F}$ of strictly positive probability density functions ϕ_{ij} (on X_j) for each couple of distinct players $i \neq j$. For every $t > 0$, they define a t-equilibrium as a Nash equilibrium of the game whose payoff of player i at $x \in X$ is $u_i(x_i, (x_j + t\varepsilon_{ij})_{j\neq i})$, where $\varepsilon_{ij} \sim \phi_{ij}$ are statistically independent random variables. Then, robust and strictly robust equilibria are defined, as in Definition 12, by considering limits of t-equilibria when t tends to zero. Existence of robust equilibria requires (1) continuity of the payoffs (2) concavity of each u_i with respect to x_i. Moreover, Andersson et al. [2] proves that robust equilibrium refines Nash equilibrium when the game is continuous.

The following proposition recalls that Prudent equilbrium refines Nash equilibrium in better-reply secure games.

[12] All these refinement concepts use perturbations (or "mistakes") of the equilibrium strategies. Other refinement concepts consider perturbations in the payoffs of the game (essential-equilibria [29] or regular-equilibria [13]). This is not connected with our paper.

[13] Andersson et al. [2] introduces a refinement notion with pure strategies, but the beliefs on other mistakes are random variables with probability density functions. Similarly, Carbonell-Nicolau ([7], [6])) extends perfect equilibrium concept to some classes of discontinuous games: the notion of robustness considered uses perturbations of players' strategies which are random variables.

Proposition 14. For every better-reply secure game, a prudent equilibrium is a Nash equilibrium.

Proof. This is a consequence of Proposition 8.

Theorem 15. *For every compact, quasiconcave and p-robust game, there exists a prudent equilibrium.*

Proof. By definition of prudent equilibrium, for every sequence $(\lambda^n)_{n\in\mathbf{N}}$ converging to 0, any limit point of Nash equilibria of G^{λ^n} (which exist from Theorem 11) is a prudent equilibrium.

Applications are given in Section 5. For example, it is proved that in first-price sealed-bid auctions with complete information, prudent equilibrium concept selects the unique natural solution, although there is a continuum of such solutions. In case of ties, if the winner is the player with the highest value (Example 27), then the game is better-reply secure and the unique prudent equilibrium is a Nash equilibrium from Proposition 14. If we now consider an equal sharing rule (Example 29), then the game is no more better-reply secure. There is no Nash equilibrium, but a familly of approximate equilibria, and the unique prudent equilibrium is the "natural" approximate Nash equilibrium of the game. Thus, Theorem 15 can refine Nash or approximate Nash equilibrium.

4 Extensions and developments

4.1 Symmetric games

The results of the previous sections can be improved upon when the game possesses enough symmetry. Following Reny [23], a game G is *symmetric*[14] if:

(1) For every players $(i,j)\in N\times N$, $X_i=X_j$. We denote $\mathbf{X}=X_1=...=X_N$.

(2) For every $(x,y)\in\mathbf{X}\times\mathbf{X}$, $u_1(x,y,...,y)=u_2(y,x,y,...,y)=...=u_N(y,...,y,x)$. We denote $v(x,y)=u_1(x,y,...,y)=...=u_N(y,...,y,x)$.

Thus, a symmetric game can be summarized by $G=(\mathbf{X},v)$.

A symmetric game $G=(\mathbf{X},v)$ is *strongly diagonally quasiconcave* (Reny [23]) if $\mathbf{X}$ is convex, and if $v(x,y)$ is quasiconcave in x. The game G is *diagonally better-reply secure* if for every (x^*,v^*) which belongs to $\overline{\{(x,v(x,x)):x\in\mathbf{X}\}}$), where $(x^*,...,x^*)$ is not a Nash equilibrium, then there exists $d\in\mathbf{X}$ and $\varepsilon>0$ such that $v(d,x')>v^*+\varepsilon$ for every $x'\in\mathbf{X}$ in some neighborhood of x^*.

Finally, Recall that a Nash equilibrium $(x_1^*,...,x_N^*)$ is symmetric if $x_1^*=...=x_N^*$.

Theorem 16. *(***Reny [23]***)*

Every symmetric, compact, diagonally quasiconcave and diagonally better-reply secure game possesses a symmetric pure Nash equilibrium.

We now adapt Definition 1 and Definition 6 to symmetric games:

Definition 17. *The symmetric game $G=(\mathbf{X},v)$ is symmetrically p-robust if for every $x\in\mathbf{X}$, for every $\varepsilon>0$ and for every neighborhood V_x of x in $\mathbf{X}$, there exists some open neighborhood $W_x\subset\mathbf{X}$ of x such that* $\sup_{x'\in V_x}\inf_{x''\in W_x} v(x',x'')\leq v(x,x)+\varepsilon$.

[14] Reny uses the terminology of "quasi-symmetric" game.

Adapting Example 4, it can be proved that every symmetric game $G = (\mathbf{X}, v)$ is symmetrically p-robust whenever it is *symmetrically piecewise u.s.c.*, which means that there exist a partition $(X^k)_{k\in K}$ of $\mathbf{X}^2$ which satisfies the two following conditions:

(1) For every $k \in K$, the multivalued functions $x \to \{x' \in \mathbf{X} : (x, x') \in X^k\}$ is lower semicontinuous.

(2) For every $k \in K$, the restriction of v to X^k is upper semicontinuous.

Definition 18. *Let $\mathcal{F}$ be the set of continuous real-valued functions c from $\mathbf{X} \times \mathbf{X}$ to $[0, +\infty]$, such that $c(x, x') = 0$ if and only if $x = x'$. Let $c \in \mathcal{F}$. For every $\lambda > 0$, the symmetric $\lambda-$prudent game associated to the symmetric game $G = (\mathbf{X}, v)$ is the symmetric game $G^{\lambda}_{sym} = (\mathbf{X}, v^{\lambda})$, where*

$$v^{\lambda}(x, x') = \inf_{x''\in X} \left\{ v_i(x, x'') + \frac{c(x'', x')}{\lambda} \right\} \tag{5}$$

A symmetric $\lambda-$equilibrium of G is a symmetric Nash equilibrium of G^{λ}. A strategy profile $x \in X$ is a symmetric prudent equilibrium of G if there exists $c \in \mathcal{F}$ such that x is the limit of symmetric λ^n-equilibria for $\lambda^n \to 0$. The strategy profile x is a symmetric strictly prudent equilibrium if this holds for any $c \in \mathcal{F}$.

Remark 19. There are two modifications with respect to the definitions of the previous sections. First, the perturbations of the other players' strategies are symmetric, which implies that the symmetric prudent game G^{λ}_{sym} is in general different from the prudent game G^{λ} (except in two-player games). Second, the sequence of λ^n-equilibria considered in Definition 18 has to be symmetric, which is a strenghtening of Definition 12. In particular, in two-player symmetric games, symmetric prudent equilibrium refines prudent equilibrium.

The proof of the following theorem is similar to this of Theorem 11 and Proposition 14.

Theorem 20. *Let $c \in \mathcal{F}$ and G be a compact, symmetric, diagonally quasiconcave and symmetrically p-robust game.*

(1) For every $\lambda > 0$, there exists a symmetric $\lambda-$equilibrium.

(2) There exists a symmetric prudent equilibrium, and for every diagonally better-reply secure game, this is a Nash equilibrium.

4.2 Beyond p-robust games: strategic approximation.

The idea of the previous sections can be extended to any quasiconcave game G as follows. For every finite subsets $X^f = \prod_{i\in N} X_i^f$ of X and every sequence $(\lambda_k)_{k\in\mathbf{N}}$ of positive reals converging to zero, we can quasiconcavify the prudent game G^{λ_k} on X^f as follows: for every player $i \in N$ and every integer $k \geq 0$, define $\tilde{u}_i^k$ on $co(X^f) = \Pi_{i=1}^N co(X_i^f)$ by

$$\tilde{u}_i^k(x_i, x_{-i}) = \sup\{\min\{u_i^{\lambda_k}(y_i^1, x_{-i}), ..., u_i^{\lambda_k}(y_i^n, x_{-i})\}\}$$

over all $n \in \mathbb{N}$ and all families $\{y_i^1, ..., y_i^n\}$ of X_i^f such that $x_i \in$co$\{y_i^1, ..., y_i^n\}$. Since X^f is finite and $u_i^{\lambda_k}$ is continuous with respect to the second argument, it is easy to see that $\tilde{u}_i^k$ is upper semicontinuous with respect to x and continuous with respect to x_{-i}. Thus, for every integer $k \geq 0$, there exists a Nash equilibrium x^k of the game $\tilde{G}^k = (co(X^f), \tilde{u}_i^k)_{i\in N}$ (see, for example, Theorem 2 in [8]). The proof of the following proposition can be found in Appendix 6.5.

Proposition 21. Let G be a quasiconcave and better-reply secure game. There exists a sequence of finite approximations $X^k \subset X$ such that any limit point of Nash equilibria of $(coX^k, \tilde{u}_i^k)_{i\in N}$ is a Nash equilibrium of G.

Following Reny [24], let us define a strategic approximation of G as "a countable subset of pure strategies with the property that limits of all equilibria of all sequences of approximating games whose finite strategy sets eventually include each member of the countable set must be equilibria of the infinite game". Thus, Proposition 21 provides a pure strategy strategic approximation of any quasiconcave and better-reply secure game, and this strategic approximation scheme is based on prudent games.

4.3 Games in mixed strategies

Denote by $M_i = \Delta(X_i)$ the set of Borel probability measures on X_i, usually called the set of mixed strategies of player i. Recall it is a compact, Hausdorff and metrizable set under the weak* topology. To every game $G = ((X_i)_{i\in N}, (u_i)_{i\in N})$, we associate its mixed strategy extension $G' = ((M_i)_{i\in N}, (\tilde{u}_i)_{i\in N})$, where $\tilde{u}_i$ is the multi-linear extension of u_i to M. Most techniques introduced in the previous sections can be applied to G'. In particular:

Corollary 22. *Let G be compact game. Then its mixed extension G' possesses a prudent equilibrium if G' is p-robust.*

A difficulty is to get sufficient conditions for p-robustness in mixed strategies. In general, p-robustness of G does not imply p-robustness of its mixed extension: consider the two player game defined by $X_1 = X_2 = [0,1]$, $u_2 = 0$, $u_1(x_1,x_2) = 0$ if $x_1 = x_2$ and $u_1(x_1,x_2) = 1$ otherwise. This game is p-robust (because it is piecewise continuous), but its mixed extension is not: indeed, consider $(\sigma_1,\sigma_2) = (0,0)$. If player 1 plays uniformly on a small neighborhood of 0, he obtains a payoff of 1, whatever the strategy of player 2. Thus, for every $\varepsilon > 0$ small enough, for every neighborhood V_{σ_2} of σ_2, and for every neighborhood V_{σ_1} of σ_1, $\sup_{\sigma_1' \in V_{\sigma_1}} \inf_{\sigma_2' \in V_{\sigma_2}} u_1(\sigma_1',\sigma_2') = 1 > u_1(\sigma_1,\sigma_2) + \varepsilon = \varepsilon$, which contradicts p-robustness of G'.

Since the main objective of this paper is to study games in pure strategies, we do not push further the case of mixed strategies. At worst, Corollary 22 can be applied to continuous games. See Example 35

5 Examples

5.1 Nash demand game

Example 23. Some amount of money can be split between two players. Each one chooses the share he demands. Then, each player receives his demand if the demand can be satisfied, and 0 otherwise. If the total amount of money is normalized at 1, the payoff of seller i is

$$u_i(x_i, x_{-i}) = \begin{cases} x_i & \text{if } x_i + x_{-i} \leq 1, \\ 0 & \textit{otherwise} \end{cases}$$

This game is compact, quasiconcave and better-reply secure. The set of Nash equilibria is $\{(1,1)\} \cup \{(x, 1-x) : x \in [0,1]\}$. It is a p-robust game (because it is piecewise continuous), thus it possesses a prudent Nash equilibrium.

Proposition 24. Assume c_1 and c_2 are distances on $[0,1] \times [0,1]$. If the relative prudence $p_{1|2}(x)$ of player 1 with respect to player 2 at every $x \in [0,1]$ is constant, equal to $\mu \in [0,+\infty]$, then the set of prudent equilibria is $\{(1,1), (\frac{1}{1+\mu}, \frac{\mu}{1+\mu})\}$.

Proof. By definition, for every $x_{-i} < 1$ and every $x_i < 1 - x_{-i}$,

$$u_i^\lambda(x_i, x_{-i}) = \inf_{y_{-i} \in X_{-i}} \{u_i(x_i, y_{-i}) + \frac{c_i(y_{-i}, x_{-i})}{\lambda}\} = \min\{u_i(x_i, x_{-i}) = x_i, \frac{c_i(1 - x_i, x_{-i})}{\lambda}\}.$$

Indeed, the infimum above can reached at $y_{-i} = x_{-i}$ or for $y_{-i} \to (1 - x_i)^+$. Moreover, for every other strategy profiles, $u_i^\lambda(x_i, x_{-i}) = 0$. If $(x_1^{\lambda_n}, x_2^{\lambda_n})$ is a sequence of Nash equilibria of G^{λ_n} which converges to (x_1, x_2) when λ_n tends to zero, then either $(x_1^{\lambda_n}, x_2^{\lambda_n}) = (1,1)$ for infinitely many $n > 0$, and at the limit, this provides the prudent equilibrium $(1,1)$. Otherwise, up to a subsequence, we can assume that $x_i^{\lambda_n} < 1$ for $i = 1,2$ and $x_1^{\lambda_n} + x_2^{\lambda_n} < 1$, otherwise one player does not play a best-response in G^{λ_n}. Moreover, since the initial game is better-reply secure, (x_1, x_2) should be a Nash equilibrium, i.e. $x_1 = 1 - x_2$. In addition, we have $\lambda_n . x_i^{\lambda_n} = c_i(1 - x_i^{\lambda_n}, x_{-i}^{\lambda_n})$ for $i = 1,2$ (thus $x_i^{\lambda_n} > 0$): indeed, otherwise, $u_i^{\lambda_n}(x)$ would be either equal to x_i or to $\frac{c_i(1-x_i, x_{-i})}{\lambda_n}$ for every x on some neighborhood of x^{λ_n}, thus player i would be able to improve its payoff in G^{λ_n} by increasing or decreasing slightly his strategy. This implies

$$\lim_{n \to +\infty} \frac{x_2^{\lambda_n}}{x_1^{\lambda_n}} = \frac{1 - x_1}{x_1} = \mu,$$

thus $(x_1, x_2) = (\frac{1}{1+\mu}, \frac{\mu}{1+\mu})$. In short, the less uncertain player obtains the bigger share.[15]

5.2 Hotelling location game

Example 25. Two sellers $i = 1,2$ sell the same good at the same price. Each seller i has to find a location along some street $x_i \in [0,1]$. Consumers are uniformly distributed on $[0,1]$, and each consumer chooses the closest seller. In case of ties ($x_i = x_{-i}$), it is assumed that the two sellers merge. In this case, the payoff of each player is assumed to increase of $c \in]0, \frac{1}{2}[$ (economies of scale, fixed cost elimination, etc.). The payoff of seller i is

$$u_i(x_i, x_{-i}) = \begin{cases} \frac{x_i + x_{-i}}{2} & \text{if } x_i < x_{-i}, \\ \frac{1}{2} + c & \text{if } x_i = x_{-i} \\ 1 - \frac{x_i + x_{-i}}{2} & \text{if } x_i > x_{-i}, \end{cases}$$

Proposition 26. The set of Nash equilibria is $\{(x,x) : x \in [\frac{1}{2} - c, \frac{1}{2} + c]\}$, and $(\frac{1}{2}, \frac{1}{2})$ is the only symmetric prudent equilibrium.

Proof. First, this game is compact, symmetric, diagonally quasiconcave and diagonally better-reply secure, second, its set of Nash equilibria is clearly $\{(x,x) : x \in [\frac{1}{2} - c, \frac{1}{2} + c]\}$. It is a symmetrically

[15] A similar idea was first proposed in [1], in a different setting.

p-robust game (because it is symmetrically piecewise continuous), thus it possesses a symmetric prudent Nash equilibrium. Let us prove that the only prudent Nash equilibrium is $(\frac{1}{2}, \frac{1}{2})$. Let (x, x) be a symmetric prudent Nash equilibrium, and assume, for example, $x > \frac{1}{2}$. By definition, (x, x) is the limit of symmetric λ_n-equilibria $(x^{\lambda_n}, x^{\lambda_n})$, where λ_n converges to zero. Remark that $v^{\lambda_n}(x^{\lambda_n}, x^{\lambda_n}) \leq 1 - x^{\lambda_n}$ (since the other player can decrease slightly its location). Moreover, for every $\varepsilon > 0$ small enough, $v^{\lambda_n}(x^{\lambda_n} - \varepsilon, x^{\lambda_n})$ converges to $v(x - \varepsilon, x) = x + \frac{\varepsilon}{2}$, which is larger than $1 - x^{\lambda_n}$ for n large enough. This contradicts that $(x^{\lambda_n}, x^{\lambda_n})$ is a λ_n-equilibria. The proof is similar when $x < \frac{1}{2}$.

5.3 First-price sealed-bid auctions

Example 27. *(First-price auction with maximum value sharing rule)*

Two bidders $i = 1, 2$ submit simultaneous sealed bids $x_i \in [0, M]$ to the seller, $M > 0$. The highest bidder wins the object and pays the value of her bid. The true values of the bidders are $v_1 < v_2 < M$. The strategy spaces are $X_1 = X_2 = [0, M]$, and the payoff of player i is defined by

$$u_i(x_i, x_{-i}) = \begin{cases} v_i - x_i & \text{if } x_i > x_{-i}, \\ 0 & \text{if } x_i < x_{-i} \end{cases}$$

Assume that in case of ties ($x_i = x_{-i}$), the winner is the bidder with the highest valuation, i.e. player 2.

Proposition 28. 1) The set of Nash equilibria is $\{(x, x) : x \in [v_1, v_2]\}$. 2) Assume that c_1 and c_2 are distances on $[0, M] \times [0, M]$ and that for every strategy profile (x, x) with $x \in]v_1, v_2]$, player 1 is infinitely more prudent than player 2 at $x \in X$. Then (v_1, v_1) is the only prudent Nash equilibrium.

First, this game is compact, quasi-concave and better-reply secure. Second, clearly, the set of Nash equilibria is the set of profiles $\{(x_1, x_1) : x_1 \in [v_1, v_2]\}$. The game is piecewise continuous (see Example 4), thus is is p-robust. Hence, there exists a prudent Nash equilibrium (Theorem 15 and Proposition 14.) Let (x, x) be a Nash equilibrium, with $x > v_1$. For player 2, a possible deviation x'_1 of player 1 does not matter if $x'_1 < x$, and $x'_1 > x$ seems unlikely (it seems more probable that player 1 will deviate for some strategy that weakly dominates x, like $x_1 = v_1$). For player 1, a deviation $x'_2 < x$ of player 2 really matters, because it implies a negative payoff $v_1 - x$, that could be avoided by playing $x_1 = v_1$. Thus, the assumption that player 1 is infinitely more prudent than player 2 at x seems reasonable. Under this assumption, we get only one prudent Nash equilibrium which is (v_1, v_1) (see the proof in appendix 6.6.)

Example 29. *(First-price auction with equal sharing rule)*

Consider the game defined in Example 27, but now assume equal sharing rule, that is $u_i(x_i, x_{-i}) = \frac{v_i - x_i}{2}$ if $x_i = x_{-i}$. There is no Nash equilibrium, because at (x, x), one player should deviate. But the game is still p-robust and quasiconcave, thus possesses a prudent equilibrium. Moreover, there is a continuum of approximate equilibria: for every $x \in [v_1, v_2]$, (x, x) is an approximate equilibrium, meaning that it is the limit of the $\frac{1}{n}$–Nash equilibrium profiles $(x, x + \frac{1}{n})$. Yet, following the proof of Proposition 28, we get:

Proposition 30. 1) The set of approximate Nash equilibria is $\{(x, x) : x \in [v_1, v_2]\}$. 2) Assume that c_1 and c_2 are distances on $[0, M] \times [0, M]$. Assume that for every strategy profile (x, x) with $x \in]v_1, v_2]$, player 1 is infinitely more prudent than player 2 at $x \in X$. Then (v_1, v_1) is the only prudent approximate Nash equilibrium.

5.4 Bertrand duopoly with symmetric costs

Example 31. Consider[16] N identical firms competing for a homogenous good. Aggregate demand is a function $D : [0, +\infty[\to [0, +\infty[$, and all firms have the same cost function $C : [0, +\infty[\to [0, +\infty[$. For every integer $m \in \{1, ...N\}$, let $v_m(p) = \frac{pD(p)}{m} - C(\frac{D(p)}{m})$ for every $p \in [0, +\infty[$. This is the profit of each of m firms which choose a same price p, when all other firms choose higher prices. Define a symmetric game as follows: firm $i \in N$ chooses a price $p_i \in X = [0, +\infty[$. Given a strategy profile $p = (p_i)_{i \in N}$, the payoff of firm i is

$$\pi_i(p) = \begin{cases} v_{card\{j: p_j = p_i\}}(p_i) & \text{if } p_i = \min\{p_1, ..., p_N\}, \\ 0 & \text{otherwise} \end{cases}$$

We assume the following:[17]

1. v_1 and v_N are continuous.
2. There exists $p^{max} > 0$ such that $v_N(p^{max}) = 0$ and $v_N(p) < 0$ for every $p > p^{max}$.
3. There exists a unique $\check{p}_N \in]0, p^{max}[$ and a unique $\hat{p}_N \in]\check{p}_N, p^{max}]$, such that $v_N(\check{p}_N) = 0$ and $v_N(\hat{p}_N) = v_1(\hat{p}_N)$.
4. There exists a unique price $\bar{p} \in]\check{p}_N, \hat{p}_N[$ such that $v_1(\bar{p}) = 0$ (price at which a monopolist makes a zero profit), where $v_1(p) > 0$ for every $p > \bar{p}$ and $v_1(p) < 0$ for every $p < \bar{p}$.
5. For every $p > \hat{p}_N$ $v_1(p) > v_N(p)$.
6. $\pi_i(p_1, p_2, ..., p_2)$ is quasiconcave in p_1 on $\mathbf{X} = [\check{p}_N, p^{\max}]$ for every $p_2 \in \mathbf{X}$.
7. For every $p_i < \check{p}_n$ and every $m \in \{1, ...N\}$, $v_m(p_i) < 0$.

From this last assumption, without any loss of generality, we can restrict the strategies to $\mathbf{X} = [\check{p}_N, p^{\max}]$. Call G the game thus defined.

In the following proposition, we assume that c is a distance on $[0, +\infty[$.

Proposition 32. (1) The set of Nash equilibria of the game G above is $\{(p, ..., p) : p \in [\check{p}_N, \hat{p}_N]\}$.

(2) $(\bar{p}, ..., \bar{p})$ is the unique symmetric prudent equilibrium of G.

Remark that this game is symmetrically p-robust, symmetric, compact, diagonally quasiconcave and diagonally better-reply secure. See the proof in Appendix 6.7.

[16] This presentation of Bertrand duopoly model follows partially [2].

[17] See [9] or [2] for natural assumptions on C and D that implies these properties.

5.5 Link with strategic uncertainty notion of Andersson et al.

These two first examples show that prudent equilibrium concept does not coincide with Andersson et al. robust equilibrium concept.

Example 33. In this example, we provide a differentiable game where Andersson et al. robustness concept refines prudent equilibrium. Consider a two-player game with $X_1 = X_2 = [0, 1]$,

$$u_1(x_1, x_2) = \begin{cases} x_1.(2x_2 - 1) & \text{if } \frac{1}{2} \le x_2 \le 1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$u_2(x_1, x_2) = -(x_1 - x_2)^2$$

It is a compact and quasiconcave game. The strategy profile $(0, 0)$ is not robust to uncertainty. Indeed, for every familly $\mathcal{F}$ of strictly positive probability density functions ϕ_{ij} (on X_j), $\varepsilon_{ij} \sim \phi_{ij}$ statistically independent random variables, and $t > 0$, we have $u_1(0, 0+t\varepsilon_{ij})_{j\neq i}) = 0$ and $u_1(1, 0+t\varepsilon_{ij})_{j\neq i}) > 0$, thus playing 0 is not optimal for player 1 in the perturbed game. Yet $(0, 0)$ is a prudent equilibria: indeed, $u^\lambda(0, 0) = 0 \ge u^\lambda(x_1, 0) = 0$ for every $x_1 \in [0, 1]$ and for $\lambda > 0$ small enough, because $u_1(0, x_2) = 0 = u_1(x_1, x_2)$ for every $x_1 \in [0, 1]$ and every x_2 in some neighborhood of 0.

Example 34. We provide an example of a game with a continuum of Nash equilibria, with only one prudent Nash equilibrium, and where every strategy profile is robust to uncertainty in the sense of Andersson et al. [2]. Consider the symmetric two-player game with $X_i = [0, 1]$ for $i = 1, 2$ and

$$u_i(x_i, x_{-i}) = \begin{cases} 0 & \text{if } x_i < x_{-i} \text{ and } x_{-i} \in \mathbb{Q}, \\ 1 & \text{otherwise} \end{cases}$$

This game is p-robust (because each u_i is u.s.c. in x_i). From Theorem 15, there exists a prudent equilibrium. Let us prove that for every $c \in \mathcal{F}$, $(1, 1)$ is the only prudent equilibrium. Clearly, (x_1, x_2) is a Nash equilibrium if and only if $u_i(x_i, x_{-i}) = 1$, $i = 1, 2$, or equivalently if $x_1 = x_2$ or $[x_1 < x_2$ and $x_2 \notin \mathbb{Q}]$ or $[x_2 < x_1$ and $x_1 \notin \mathbb{Q}]$. Moreover, for every $c \in \mathcal{F}$ and every $\lambda > 0$, $u_i^\lambda(x_i, x_{-i}) = 0$ for every $x_i \le x_{-i}$ if $x_{-i} < 1$. Indeed, under these conditions,

$$u_i^\lambda(x_i, x_{-i}) = \inf_{y_{-i} \in X_{-i}} \{u_i(x_i, y_{-i}) + \frac{c_i(y_{-i}, x_{-i})}{\lambda}\} \le u_i(x_i, x_{-i}+\varepsilon_n) + \frac{c_i(x_{-i} + \varepsilon_n, x_{-i})}{\lambda} = \frac{c_i(x_{-i} + \varepsilon_n, x_{-i})}{\lambda}$$

where $\varepsilon_n > 0$ can be chosen so that $x_{-i}+\varepsilon_n \in \mathbb{Q}\cap[0, 1]$, $x_{-i}+\varepsilon_n > x_i$ and such that $(\varepsilon_n)_{n\in\mathbf{N}}$ converges to 0. Passing to the limit as $\varepsilon_n \to 0$, we get $u_i^\lambda(x_i, x_{-i}) = 0$ from the continuity of c and $c(x_{-i}, x_{-i}) = 0$. Moreover, $u_i^\lambda(1, x_{-i}) = 1$ for every $x_{-i} \in [0, 1]$, thus a best reply of player i to a strategy $x_{-i} < 1$ is strictly larger than x_{-i}, and $x_i = 1$ is the unique best-reply to $x_{-i} = 1$ in G^λ. Thus, the unique Nash equilibrium of G^λ is $(1, 1)$, and $(1, 1)$ is finally a (strictly) prudent equilibrium, and the only one.

Now, for every familly $\mathcal{F}$ of strictly positive probability density functions ϕ_{ij} (on X_j), $\varepsilon_{ij} \sim \phi_{ij}$ statistically independent random variables, and $t > 0$, we have $u_i(x_i, (x_j + t\varepsilon_{ij})_{j\neq i}) = 1$. Thus every $(x_i, x_{-i}) \in [0, 1] \times [0, 1]$ is strictly robust in the sense of Andersson et al. (even if x is not a Nash equilibrium).

5.6 Example of a prudent equilibrium in mixed strategies

Example 35. Consider the following two-player finite game:

$$\begin{array}{ccc} & A & B \\ a & (0,0) & (-1,0) \\ b & (0,-1) & (0,0) \end{array}$$

Let G be the mixed extension of this game. Every strategy of player 1 (resp. of player 2) can be written $(\lambda, 1-\lambda) \in [0,1]^2$, where λ is the probability of a (resp. A), and $1-\lambda$ is the probability of b (resp. B). Assume that c is quadratique, i.e. more precisely that $c((\lambda, 1-\lambda), (\lambda', 1-\lambda')) = (\lambda - \lambda')^2$. There are two Nash equilibria which are (a, A) and (b, B). Only (b, B) is perfect.

Let us prove that (a, A) is not a prudent equilibrium. Indeed, by contradiction, assume that (σ_1^n, σ_2^n) is a Nash equilibrium of G^{λ^n} for $\lambda^n \to 0$, where (σ_1^n, σ_2^n) converges to $(a, A) = ((1,0),(1,0))$. Then, $u_1^{\lambda_n}(\sigma_1^n, \sigma_2^n) \geq u_1^{\lambda_n}(b, \sigma_2^n) = 0$ which is only possible if $\sigma_2^n = A$, and by a symmetric argument $\sigma_1^n = a$. But then, by definition of $u_1^{\lambda_n}$,

$$u_1^{\lambda_n}(a, A) \leq u_1^{\lambda_n}(a, \alpha_n B + (1-\alpha_n)A) + \frac{c(\alpha_n B + (1-\alpha_n)A, A)}{\lambda^n} = -\alpha_n + \frac{\alpha_n^2}{\lambda^n}$$

where $\alpha_n > 0$ can be chosen small enough for every $n > 0$ such that $-\alpha_n + \frac{\alpha_n^2}{\lambda^n}$ is negative. Then we get $u_1^{\lambda_n}(a, A) < 0 = u_1^{\lambda_n}(b, A)$. But this contradicts that (a, A) is a Nash equilibrium of G^{λ_n}.

6 Appendix

6.1 Example 4: a piecewise upper semicontinuous game is p-robust

Let G be a piecewise upper semicontinuous game. We first prove the following claim:

Claim 36. *For every open neighborhood $V_{x_{-i}}$ of $x_{-i} \in X_{-i}$, $\inf_{x'_{-i} \in V_{x_{-i}}} u_i(x_i, x'_{-i})$ is upper semicontinuous with respect to x_i.*

Proof. Let $a \in \mathbb{R}$, $V_{x_{-i}}$ be an open neighborhood of $x_{-i} \in X_{-i}$, and $(x_i^n)_{n \in \mathbf{N}}$ be a sequence of strategy profiles converging to $x_i \in X$, and such that for every integer n,

$$\inf_{x'_{-i} \in V_{x_{-i}}} u_i(x_i^n, x'_{-i}) \geq a.$$

For every $k \in K_i$, let $X_{-i}^{k,i}(x_i) = \{x_{-i} \in X_{-i} : (x_i, x_{-i}) \in X^{k,i}\}$. Let $k \in K_i$ such that $x_{-i} \in X_{-i}^{k,i}(x_i)$. Since $X_{-i}^{k,i} \cap V_{x_{-i}}$ is a lower semicontinuous multivalued function (since $X_{-i}^{k,i}$ is lower semicontinuous and $V_{x_{-i}}$ is open), considering a subsequence of $(x_i^n)_{n \in \mathbf{N}}$ if necessary, there exists a sequence x_{-i}^n converging to x_{-i} such that $x_{-i}^n \in X_{-i}^{k,i}(x_i^n) \cap V_{x_{-i}}$. In particular, $u_i(x_i^n, x_{-i}^n) \geq a$ for n large enough. Passing to the limit, from upper semicontinuity of the restriction of u_i to $X^{k,i}$, we get $u_i(x_i, x_{-i}) \geq a$. Passing to the infimum with respect to $x_{-i} \in V_{x_{-i}}$, we get finally

$$\inf_{x'_{-i} \in V_{x_{-i}}} u_i(x_i, x'_{-i}) \geq a$$

which ends the proof of the claim.

Now, to prove that G is p-robust, consider $\varepsilon > 0$ and let $V_{x_{-i}}$ be an open neighborhood of x_{-i}. From the claim above, there exists V_{x_i} such that for every $x'_i \in V_{x_i}$,

$$\inf_{x'_{-i} \in V_{x_{-i}}} u_i(x'_i, x'_{-i}) \leq \inf_{x'_{-i} \in V_{x_{-i}}} u_i(x_i, x'_{-i}) + \varepsilon \leq u_i(x) + \varepsilon,$$

which proves p-robustness.

6.2 Proof of Proposition 7

(1) Remark that $u_i^\lambda(x_i, x_{-i})$ is the infimum of a family of functions which are quasiconcave in x_i, thus it is quasiconcave in x_i.

(2) First note that $u_i^\lambda(x_i, x_{-i})$ is the infimum of a family of functions which are continuous in x_{-i}. Thus it is upper semicontinuous in x_{-i}. To prove it is lower semicontinuous in x_{-i}, consider a sequence $(x^n_{-i})_{n \in \mathbf{N}}$ converging to some x_{-i}, and such that $u_i^\lambda(x_i, x^n_{-i}) \leq \alpha$ for some real α and for every integer $n \geq 0$. By definition,

$$\inf_{y_{-i} \in X_{-i}} \{u_i(x_i, y_{-i}) + \frac{c_i(y_{-i}, x^n_{-i})}{\lambda}\} \leq \alpha$$

for every integer n. Given $\varepsilon > 0$, this implies that there is a sequence $y^n_{-i} \in X_{-i}$ such that

$$u_i(x_i, y^n_{-i}) + \frac{c_i(x^n_{-i}, y^n_{-i})}{\lambda} \leq \alpha + \varepsilon.$$

Since c_i is continuous on the compact set $X_{-i} \times X_{-i}$, it is uniformly continuous, thus

$$u_i(x_i, y^n_{-i}) + \frac{c_i(x_{-i}, y^n_{-i})}{\lambda} \leq \alpha + 2\varepsilon$$

for n large enough. Passing to the infimum with respect to the second variable y^n_{-i}, we get $u_i^\lambda(x_i, x_{-i}) \leq \alpha + 2\varepsilon$. Since this is true for every $\varepsilon > 0$, this finally proves (2).

(3) Assume G is p-robust, and prove that u_i^λ is u.s.c. in x. Take $a \in \mathbb{R}$ and consider a sequence $(x^n)_{n \in \mathbf{N}}$ of strategy profiles converging to $x \in X$, and such that $u_i^\lambda(x^n) \geq a$ for every integer n. We have to prove that $u_i^\lambda(x) \geq a$. By definition of $(x^n)_{n \in \mathbf{N}}$, for every integer $n \in \mathbf{N}$

$$\inf_{y_{-i} \in X_{-i}} \left\{u_i(x^n_i, y_{-i}) + \frac{c_i(y_{-i}, x^n_{-i})}{\lambda}\right\} \geq a.$$

Let $\varepsilon > 0$ and $y_{-i} \in X_{-i}$. By p-robustness, for every integer $k \in \mathbf{N}$, choosing $V_{y_{-i}} = B(y_{-i}, \frac{1}{k})$, there is some open neighborhood $V^k_{x_i}$ of x_i such that: for every $x'_i \in V^k_{x_i}$, there exists $y^k_{-i} \in B(y_{-i}, \frac{1}{k})$ such that

$$u(x'_i, y^k_{-i}) \leq u_i(x_i, y_{-i}) + \varepsilon$$

In particular, for every k, there is n_k large enough such that for every $n \geq n_k$, there is $y^n_{-i} \in B(y_{-i}, \frac{1}{k})$ such that

$$u_i(x^n_i, y^n_{-i}) \leq u_i(x_i, y_{-i}) + \varepsilon$$

Thus

$$u_i(x^n_i, y^n_{-i}) + \frac{c_i(y^n_{-i}, x^n_{-i})}{\lambda} \leq u_i(x_i, y_{-i}) + \frac{c_i(y_{-i}, x_{-i})}{\lambda} + (\frac{c_i(y^n_{-i}, x^n_{-i})}{\lambda} - \frac{c_i(y_{-i}, x_{-i})}{\lambda}) + \varepsilon,$$

From continuity of c_i, for k large enough and $n \geq n_k$, we get

$$u_i(x_i^n, y_{-i}^n) + \frac{c_i(y_{-i}^n, x_{-i}^n)}{\lambda} \leq u_i(x_i, y_{-i}) + \frac{c_i(y_{-i}, x_{-i})}{\lambda} + 2\varepsilon$$

Passing to the infimum with respect to y_{-i} in the right-hand side, then to the infimum with respect to y_{-i}^n in the left-hand side, we get, for n large enough:

$$a \leq u_i^\lambda(x^n) \leq u_i^\lambda(x) + 2\varepsilon.$$

Consequently, u_i^λ is an u.s.c. function of x

For the last point, take $x \in X$, and consider a sequence $(x_{-i}^n)_{n\in\mathbf{N}}$ converging to x_{-i} such that $\underline{u}_i(x) = \lim_{n\to+\infty} u_i(x_i, x_{-i}^n)$. By definition

$$u_i^\lambda(x_i, x_{-i}) = \inf_{y_{-i}\in X_{-i}} \{u_i(x_i, y_{-i}) + \frac{c_i(y_{-i}, x_{-i})}{\lambda}\} \leq u_i(x_i, x_{-i}^n) + \frac{c_i(x_{-i}^n, x_{-i})}{\lambda}.$$

Passing to the limit, we get $u_i^\lambda(x_i, x_{-i}) \leq \underline{u}_i(x_i, x_{-i})$.

6.3 Proof of Proposition 8

(1) Let $(\lambda_n)_{n\in\mathbf{N}}$ be a sequence of non negative reals converging to $+\infty$, and $(x^n)_{n\in\mathbf{N}}$ be a sequence of λ^n-equilibria converging to $x \in X$. By definition of a Nash equilibrium in the game G^{λ_n}, we get

$$\forall i \in N,\ \forall d_i \in X_i,\ \inf_{y_{-i}\in X_{-i}} \{u_i(d_i, y_{-i}) + \frac{c_i(y_{-i}, x_{-i}^n)}{\lambda_n}\} \leq \inf_{y_{-i}\in X_{-i}} \{u_i(x_i^n, y_{-i}) + \frac{c_i(y_{-i}, x_{-i}^n)}{\lambda_n}\} \tag{6}$$

Since c_i is continuous and X is compact, for every $\varepsilon > 0$, there exists n large enough such that

$$\forall i \in N,\ \forall d_i \in X_i,\ \inf_{y_{-i}\in X_{-i}} \{u_i(d_i, y_{-i})\} \leq \inf_{y_{-i}\in X_{-i}} \{u_i(x_i^n, y_{-i})\} + \varepsilon \tag{7}$$

i.e. x_i^n is a $\varepsilon-$maximin strategy for player i for n large enough. From Claim 36, $\inf_{y_{-i}\in X_{-i}}\{u_i(x_i, y_{-i})\}$ is u.s.c. with respect to x_i. Thus, passing to the limit when $n \to +\infty$ and $\varepsilon \to 0$ in the above inequality, we get

$$\forall i \in N,\ \forall d_i \in X_i,\ \inf_{y_{-i}\in X_{-i}} \{u_i(d_i, y_{-i})\} \leq \inf_{y_{-i}\in X_{-i}} \{u_i(x_i, y_{-i})\} \tag{8}$$

that is x is a maximin equilibrium.

(2) We will use the following Claim, which lists some additional important properties of u_i^λ.

Claim 37. *Let $G = ((X_i)_{i\in N}, (u_i)_{i\in N})$ be a game, and $G^\lambda = ((X_i)_{i\in N}, (u_i^\lambda)_{i\in N})$ the λ-prudent game associated to G.*

1. *If $x_{-i}^n \to x_{-i}$ and $\lambda_n \to 0$ then for every $x_i \in X_i$, $\liminf_{n\to+\infty} u_i^{\lambda_n}(x_i, x_{-i}^n) \geq \underline{u_i}(x_i, x_{-i})$.*
2. *For every $x \in X$, $u_i^{\lambda_n}(x_i, x_{-i})$ tends to $\underline{u}_i(x_i, x_{-i})$ when λ_n tends to 0.*
3. *$u_i^\lambda(x_i, x_{-i}) = \inf_{y_{-i}\in X_{-i}}\{\underline{u}_i(x_i, y_{-i}) + \frac{c_i(x_{-i}, y_{-i})}{\lambda}\}$*

Proof of Claim 37. For the first point of the claim, let $x_{-i}^n \to x_{-i}$ and consider a sequence $(\lambda_n)_{n\in\mathbf{N}}$ converging to 0. By definition,

$$u_i^{\lambda_n}(x_i, x_{-i}^n) = \inf_{y_{-i}\in X_{-i}} \{u_i(x_i, y_{-i}) + \frac{c_i(y_{-i}, x_{-i}^n)}{\lambda_n}\}.$$

Let $\varepsilon > 0$. By definition of infimum, there is a sequence $y^n_{-i} \in X_{-i}$ such that

$$u_i^{\lambda_n}(x_i, x^n_{-i}) \geq u_i(x_i, y^n_{-i}) + \frac{c_i(x^n_{-i}, y^n_{-i})}{\lambda_n} - \varepsilon.$$

Since the sequence $u_i^{\lambda_n}(x_i, x^n_{-i})$ is bounded, this implies that the sequence $\frac{c_i(x^n_{-i}, y^n_{-i})}{\lambda_n}$ is bounded, thus y^n_{-i} converges to x_{-i} (because λ_n converges to 0). Moreover, since $c_i \geq 0$, we get

$$u_i^{\lambda_n}(x_i, x^n_{-i}) \geq u_i(x_i, y^n_{-i}) - \varepsilon,$$

and passing to the infimum limit as $n \to +\infty$, then taking $\varepsilon \to 0$, we get 1.

For the second point of the claim, use the first property with a constant sequence $x^n_{-i} = x_{-i}$, and Point 4 in Proposition 7.

For the last point of the claim, first note that the inequality $u_i^{\lambda}(x_i, x_{-i}) \geq \inf_{y_{-i} \in X_{-i}} \{\underline{u}_i(x_i, y_{-i}) + \frac{c_i(x_{-i}, y_{-i})}{\lambda}\}$ is obvious. To prove the converse inequality, let $\varepsilon > 0$ and $\bar{y}_{-i} \in X_{-i}$ such that $\inf_{y_{-i} \in X_{-i}} \{\underline{u}_i(x_i, y_{-i}) + \frac{c_i(x_{-i}, y_{-i})}{\lambda}\} \geq \underline{u}_i(x_i, \bar{y}_{-i}) + \frac{c_i(x_{-i}, \bar{y}_{-i})}{\lambda} - \varepsilon$. By definition of $\underline{u}_i$ and from the continuity of c_i, there exists a sequence $y^n_{-i} \in X_{-i}$, converging to $\bar{y}_{-i}$, such that $\underline{u}_i(x_i, \bar{y}_{-i}) + \frac{c_i(x_{-i}, \bar{y}_{-i})}{\lambda} - \varepsilon \geq u_i(x_i, y^n_{-i}) + \frac{c_i(x_{-i}, y^n_{-i})}{\lambda} - 2\varepsilon$ for n large enough. Passing to the infimum with respect to the second variable in the right-hand side, we finally get $\inf_{y_{-i} \in X_{-i}} \{\underline{u}_i(x_i, y_{-i}) + \frac{c_i(x_{-i}, y_{-i})}{\lambda}\} \geq u_i^{\lambda}(x_i, x_{-i}) - 2\varepsilon$, which ends the proof of the Claim.

Now, we prove the second part of Proposition 8. Assume G is a better reply secure game, let λ_n be a sequence of non negative reals converging to 0, and x^n be a sequence of λ_n-equilibria which converges to x. From compacity of Γ, without any loss of generality, up to a subsequence, we can assume that $(x^n, u(x^n))$ converges to some $(x, v) \in \Gamma$. By definition,

$$\forall i \in N,\ \forall d_i \in X_i,\ u_i^{\lambda_n}(d_i, x^n_{-i}) \leq u_i^{\lambda_n}(x^n) \leq u_i(x^n) \tag{9}$$

the last inequality being a consequence of Proposition 7. Passing to the infimum limit as $n \to +\infty$, and using Point (1) in the above claim, we get $\underline{u}_i(d_i, x_{-i}) \leq u_i$ for every $i \in N$ and every $d_i \in X_i$. Since G is better reply secure, this implies that x is a Nash equilibrium.

6.4 Proof of Proposition 10

First, assume that v_i satisfies Equation 3. For every $(x, y_{-i}, z_{-i}) \in X \times X_{-i} \times X_{-i}$, from Equation 3, and since c_i is a distance, we get

$$v_i(x_i, x_{-i}) \leq u_i(x_i, z_{-i}) + \frac{c_i(z_{-i}, x_{-i})}{\lambda} \leq u_i(x_i, z_{-i}) + \frac{c_i(z_{-i}, y_{-i})}{\lambda} + \frac{c_i(y_{-i}, x_{-i})}{\lambda}. \tag{10}$$

Passing to the infimum with respect to z_{-i} in Equation 10, we get

$$v_i(x_i, x_{-i}) \leq v_i(x_i, y_{-i}) + \frac{c_i(y_{-i}, x_{-i})}{\lambda},$$

thus v_i satisfies Equation 4.

For the converse implication, we now assume that v_i satisfies Equation 4, and we will prove that Equation 3 is true with $u_i = v_i$. By definition, we have

$$v_i(x_i, x_{-i}) \geq \inf_{y_{-i} \in X_{-i}} \{v_i(x_i, y_{-i}) + \frac{c_i(y_{-i}, x_{-i})}{\lambda}\}$$

since we can take $y_{-i} = x_{-i}$ in the infimum and since $c(x_{-i}, x_{-i}) = 0$. For the converse inequality, remark that from Equation 4, we have

$$v_i(x_i, y_{-i}) + \frac{c_i(y_{-i}, x_{-i})}{\lambda} \geq v_i(x_i, x_{-i})$$

for every $y_{-i} \in X_{-i}$, and passing to the infimum with respect to y_{-i} we finally get

$$v_i(x_i, x_{-i}) = \inf_{y_{-i} \in X_{-i}} \left\{ v_i(x_i, y_{-i}) + \frac{c_i(y_{-i}, x_{-i})}{\lambda} \right\}$$

6.5 Proof of Proposition 21

We first prove the following lemma:

Lemma 38. If G is quasiconcave and better-reply secure, then there exists a countable set $\prod_{i=1}^N X_i' \subset X$ such that for every $(x^*, u^*) \in \Gamma$, if x^* is not a Nash equilibrium of G, then there exists $i \in N$ and $d_i \in X_i'$ such that $\underline{u_i}(d_i, x^*_{-i}) > u^*_i$.

Proof of Lemma 38. Let $\Gamma^{equ} = \{(x^*, u^*) \in \Gamma,\ x^* \text{ is a Nash equilibrium}\}$ and $\Gamma^{nequ} = \{(x^*, u^*) \in \Gamma,\ x^* \text{ is not a Nash equilibrium}\} =^c \Gamma_{equ}$. For every $(x^*, u^*) \in \Gamma^{nequ}$, from better-reply security, there exists some player i and some strategy $d \in X_i$ such that $u^*_i < \underline{u}_i(d, x^*_{-i})$. Since by definition $\underline{u}_i$ is l.s.c. with respect to x_{-i}, there exists an open neighborhood $V_{(x^*,u^*)}(d)$ of (x^*, u^*) in Γ^{nequ} such that for every $(x, u) \in V_{(x^*,u^*)}(d)$, $u_i < \underline{u}_i(d, x_{-i})$. Since Γ is a compact subset of a metric space, it is separable. Thus Γ^{nequ}, as a subset of a separable metric space, is separable. Thus, it is a Lindelöf space[18], i.e. every open cover of Γ^{nequ} has a countable subcover. Consequently, there exists a countable covering $\mathcal{O}$ of Γ^{nequ} by some open neighborhoods $V_{x^*(j),u^*(j)}(d(j))$, where $(x^*(j), u^*(j)) \in \Gamma^{nequ}$, $d(j) \in \cup_{i=1}^N X_i$ and $j \in \mathbb{N}$. Now, define $X_i' = \{d(j),\ j \in \mathbb{N}\} \cap X_i$ if it is nonempty, and X_i' be any point of X_i otherwise. By construction, it satisfies the conclusion of Lemma above.

To prove Proposition 21, consider an increasing sequence of finite subsets $X^k = \prod_{i=1}^N X_i^k$ of X such that $\cup_k X^k = X'$ (X' being defined in the above lemma) and take a sequence $(x^k)_{k \in \mathbf{N}}$ of Nash equilibria of the games $(coX^k, \tilde{u}_i^k)_{i \in N}$. By compactness of X, without any loss of generality, we can suppose that x^k converges to $x^* \in X$. By definition of $\tilde{u}_i^k$ and from Point (4) in Proposition 7, we have

$$\tilde{u}_i^k(x_i^k, x_{-i}^k) = \sup\{\min\{u_i^{\lambda_k}(y_i^1, x_{-i}^k), ..., u_i^{\lambda_k}(y_i^n, x_{-i}^k)\}\} \leq \sup\{\min\{u_i(y_i^1, x_{-i}^k), ..., u_i(y_i^n, x_{-i}^k)\}\},$$

the supremum being taken over all $n \in \mathbb{N}$ and all families $\{y_i^1, ..., y_i^n\}$ of X_i^f such that $x_i^k \in \text{co}\{y_i^1, ..., y_i^n\}$. From quasiconcavity of u_i with respect to x_i, we finally get $\tilde{u}_i^k(x_i^k, x_{-i}^k) \leq u_i(x_i^k, x_{-i}^k)$. In addition, the definition of $\tilde{u}_i^k$ gives $u_i^{\lambda_k}(d_i, x_{-i}^k) \leq \tilde{u}_i^k(d_i, x_{-i}^k)$ for every $d_i \in X_i^k$.

Now, fix $d_i \in X_i'$. For $k > 0$ large enough, $d_i \in X_i^k$, and by definition of x^k we get

$$u_i^{\lambda_k}(d_i, x_{-i}^k) \leq \tilde{u}_i^k(d_i, x_{-i}^k) \leq \tilde{u}_i^k(x^k) \leq u_i(x^k).$$

Passing to the limit as $k \to +\infty$, and from Point (1) in Claim 37, we get

$$\forall d_i \in X_i',\ \underline{u}_i(d_i, x^*_{-i}) \leq u^*_i$$

where $(x^*, u^*) \in \Gamma$. This proves that x^* is a Nash equilibrium by construction of X'.

[18] Let X be a separable metric space (which means that there exists C, a countable and dense subset of X). Then X is a Lindelöf space, i.e. every open cover of X has a countable subcover.

6.6 Proof of Proposition 28

We want to prove that (v_1, v_1) is the unique prudent equilibrium. Assume by contradiction that there is a sequence of λ_n-equilibria $x^{\lambda_n} = (x_1^{\lambda_n}, x_2^{\lambda_n})$ which converges to (x, x) when $\lambda_n \to 0^+$, where $x \in]v_1, v_2]$. If $x_1^{\lambda_n} \geq x_2^{\lambda_n}$ then $\underline{u}_1(x^{\lambda_n}) = v_1 - x_1^{\lambda_n}$ for n large enough. Thus from (4) in Proposition 7, $u_1^{\lambda_n}(x^{\lambda_n}) \leq v_1 - x_1^{\lambda_n} < 0$ for n large enough. But, by definition, x^{λ_n} is a Nash of G^{λ_n}, thus for n large enough, we get $0 = u_1^{\lambda_n}(0, x_2^{\lambda_n}) \leq u_1^{\lambda_n}(x^{\lambda_n}) < 0$, a contradiction. Now, we can assume that $x_1^{\lambda_n} < x_2^{\lambda_n}$ for n large enough. By definition,

$$u_2^{\lambda_n}(x^{\lambda_n}) = \inf_{y_1 \in [0,M]} \{u_2(y_1, x_2^{\lambda_n}) + \frac{c_2(y_1, x_1^{\lambda_n})}{\lambda_n}\},$$

or equivalently

$$u_2^{\lambda_n}(x^{\lambda_n}) = \min\left\{u_2(x^{\lambda_n}) = v_2 - x_2^{\lambda_n}, \inf_{y_1 > x_2^{\lambda_n}} \{u_2(y_1, x_2^{\lambda_n}) + \frac{c_2(y_1, x_1^{\lambda_n})}{\lambda_n} = \frac{c_2(y_1, x_1^{\lambda_n})}{\lambda_n}\}\right\}$$

because $u_2(x^{\lambda_n}) = v_2 - x_2^{\lambda_n} < u_2(y_1, x_2^{\lambda_n}) + \frac{c_2(y_1, x_1^{\lambda_n})}{\lambda_n} = v_2 - x_2^{\lambda_n} + \frac{c_2(y_1, x_1^{\lambda_n})}{\lambda_n}$ for every $0 \leq y_1 \leq x_2^{\lambda_n}$ and $y_1 \neq x_1^{\lambda_n}$. Remark also that $\inf_{y_1 > x_2^{\lambda_n}} \{\frac{c_2(y_1, x_1^{\lambda_n})}{\lambda_n}\} = \frac{c_2(x_2^{\lambda_n}, x_1^{\lambda_n})}{\lambda_n}$ because c_2 is a distance. Thus finally,

$$u_2^{\lambda_n}(x^{\lambda_n}) = \min\{v_2 - x_2^{\lambda_n}, \frac{c_2(x_2^{\lambda_n}, x_1^{\lambda_n})}{\lambda_n}\}.$$

By continuity, this equality is true on some open neighborhood of x^{λ_n}. If $v_2 - x_2^{\lambda_n} < \frac{c_2(x_2^{\lambda_n}, x_1^{\lambda_n})}{\lambda_n}$, then we get $u_2^{\lambda_n}(x_1^{\lambda_n}, x_2') = v_2 - x_2'$ for x_2' close enough to $x_2^{\lambda_n}$ which contradicts that $x_2^{\lambda_n}$ is a best reply of player 2 (in G^{λ_n}) to $x_1^{\lambda_n}$. Thus, for n large enough, we can assume $v_2 - x_2^{\lambda_n} \geq \frac{c_2(x_2^{\lambda_n}, x_1^{\lambda_n})}{\lambda_n}$. In particular, the sequence $\frac{c_2(x_2^{\lambda_n}, x_1^{\lambda_n})}{\lambda_n}$ is bounded. Since player 1 is infinitely more prudent than player 2 at x, we get

$$\lim_{n \to +\infty} \frac{c_1(x_2^{\lambda_n}, x_1^{\lambda_n})}{\lambda_n} = 0.$$

Recalling that $x_1^{\lambda_n}$ is a best reply to $x_2^{\lambda_n}$ for player 1, we get

$$0 = u_1^{\lambda_n}(0, x_2^{\lambda_n}) \leq u_1^{\lambda_n}(x^{\lambda_n}) \leq u_1(x_1^{\lambda_n}, x_1^{\lambda_n} - \varepsilon_n) + \frac{c_1(x_1^{\lambda_n} - \varepsilon_n, x_2^{\lambda_n})}{\lambda_n} = v_1 - x_1^{\lambda_n} + \frac{c_1(x_1^{\lambda_n} - \varepsilon_n, x_2^{\lambda_n})}{\lambda_n}, \quad (11)$$

the last inequality being a consequence of the definition of $u_1^{\lambda_n}(x^{\lambda_n})$. Here, $\varepsilon_n > 0$ can be chosen small enough (for n large enough), such that $| \frac{c_1(x_1^{\lambda_n} - \varepsilon_n, x_2^{\lambda_n})}{\lambda_n} - \frac{c_1(x_1^{\lambda_n}, x_2^{\lambda_n})}{\lambda_n} | \leq \frac{1}{n}$, which guarantees $\lim_{n \to +\infty} \frac{c_1(x_1^{\lambda_n} - \varepsilon_n, x_2^{\lambda_n})}{\lambda_n} = 0$. Passing to the limit in Equation 11, we get $0 \leq v_1 - x$, a contradiction. $\square$

6.7 Proof of Proposition 32

(1) can be found in Dastidar [9].

(2) Let

$$v(p_1, p_2) = \begin{cases} v_N(p_1) & \text{if } p_1 = p_2, \\ v_1(p_1) & \text{if } p_1 < p_2, \\ 0 & \text{otherwise} \end{cases}$$

and $\mathbf{X} = [\check{p}_N, p^{\max}]$. By definition, the game $G = (\mathbf{X}, v)$ is symmetric, since $\pi_i(p_i, p_2, ..., p_2) = v(p_i, p_2)$ for every $(p_i, p_2) \in \mathbf{X}^2$. To prove that G is symmetrically p-robust, we can prove that it is symmetrically piecewise continuous (see the characterizaton after Definition 17). Let us define the following partition of $\mathbf{X}^2$: $X^1 =$

$\{(p,p') \in \mathbf{X}^2 : p > p'\}$, $X^2 = \{(p,p') \in \mathbf{X}^2 : p < p'\}$ and $X^3 = \{(p,p') \in \mathbf{X}^2 : p = p'\}$. For each $k \in \{1,2,3\}$, the multivalued mapping $p \to \{p' \in \mathbf{X} : (p,p') \in X^k\}$ from $\mathbf{X}$ to $\mathbf{X}$ is clearly lower semicontinuous, and the restrictions of v to each X^k are continuous, because v_N and v_1 are continuous. In addition, he game is strongly diagonally quasiconcave when the strategies are restricted to $[\check{p}_N, p^{max}]$ by assumption.

To prove that G is *diagonally better-reply secure*, let (p^*, v^*) in $\overline{\{(p, v(p,p)) : p \in \mathbf{X}\}}$), where $(p^*, ..., p^*)$ is not a Nash equilibrium (thus $p^* > \hat{p}_N$). Since v_N is continuous, $v^* = v_N(p^*)$. But then, any $p^* - \varepsilon$ for $\varepsilon > 0$ small enough, secures strictly a payoff above v^*, because from the assumptions of the model, $v(p^* - \varepsilon, p^*) = v_1(p^* - \varepsilon) > v_N(p^*) = v^*$ for $\varepsilon > 0$ small enough (indeed, by assumption, for every $p > \hat{p}_N$, we have $v_1(p) > v_N(p)$), and from the continuity of v_1, this inequality is robust to a small modification of the other players' strategies.

Now, applying Theorem 20, for every $c \in \mathcal{F}$, there exists a symmetric prudent equilibrium $(p, ...p)$, which is also a symmetric Nash equilibrium (which implies $p \in [\check{p}_N, \hat{p}_N]$).

(3) We want to prove uniqueness, and more precisely that for every $c \in \mathcal{F}$, if $(p, ...p)$ is a symmetric prudent equilibrium, then $p = \bar{p}$. Assume first that $p > \bar{p}$. By definition, there exists a sequence of positive reals $(\lambda_n)_{n \in \mathbf{N}}$ converging to 0, and a sequence of symmetric equilibria $(p^{\lambda_n}, ..., p^{\lambda_n})$ of $G^{\lambda_n}_{sym}$, which converges to $(p, ..., p)$. Thus

$$\forall d \in \mathbf{X},\ v^{\lambda_n}(d, p^{\lambda_n}) \leq v^{\lambda_n}(p^{\lambda_n}, p^{\lambda_n}) \tag{12}$$

Recall that by definition,

$$v^{\lambda_n}(p^{\lambda_n}, p^{\lambda_n}) = \inf_{p' \in X} \left\{ v(p^{\lambda_n}, p') + \frac{c(p', p^{\lambda_n})}{\lambda_n} \right\}.$$

For every $p' < p^{\lambda_n}$, $v(p^{\lambda_n}, p') = 0$. Thus, taking $p' \to (p^{\lambda_n})^-$, we get, from the continuity of c and from equation 12:

$$\forall d \in \mathbf{X},\ v^{\lambda_n}(d, p^{\lambda_n}) \leq v^{\lambda_n}(p^{\lambda_n}, p^{\lambda_n}) \leq 0. \tag{13}$$

Now, we claim that there exists n large enough and $\varepsilon > 0$ such that $p^{\lambda_n} - \varepsilon > \bar{p}$, and such that for every $p' \leq p^{\lambda_n} - \varepsilon$, we have $\frac{c(p^{\lambda_n}, p')}{\lambda_n} > v^{\lambda_n}(p^{\lambda_n} - \varepsilon, p^{\lambda_n}) + 1$. This is possible because otherwise, up to a subsequence, we would be able to build a sequence $p'_n \leq p^{\lambda_n} - \varepsilon$ such that $\frac{c(p^{\lambda_n}, p'_n)}{\lambda_n} \leq v^{\lambda_n}(p^{\lambda_n} - \varepsilon, p^{\lambda_n}) + 1$. But $v^{\lambda_n}(p^{\lambda_n} - \varepsilon, p^{\lambda_n}) + 1$ is bounded, and $\frac{c(p^{\lambda_n}, p'_n)}{\lambda_n}$ tends to $+\infty$, a contradiction.

By definition,

$$v^{\lambda_n}(p^{\lambda_n} - \varepsilon_n, p^{\lambda_n}) = \inf_{p' \in \mathbf{X}} \left\{ v(p^{\lambda_n} - \varepsilon_n, p') + \frac{c(p', p^{\lambda_n})}{\lambda_n} \right\}$$

Recall that from our choice above, if $p' \leq p^{\lambda_n} - \varepsilon$ then $v(p^{\lambda_n} - \varepsilon_n, p') + \frac{c(p', p^{\lambda_n})}{\lambda_n} > v^{\lambda_n}(p^{\lambda_n} - \varepsilon, p^{\lambda_n}) + 1$, thus in particular

$$v^{\lambda_n}(p^{\lambda_n} - \varepsilon_n, p^{\lambda_n}) = \inf_{p' > p^{\lambda_n} - \varepsilon} \left\{ v_1(p^{\lambda_n} - \varepsilon) + \frac{c(p^{\lambda_n}, p')}{\lambda_n} \right\}$$

and finally, since c is a distance, this minimum is reached for $p' = p^{\lambda_n}$, that is

$$v^{\lambda_n}(p^{\lambda_n} - \varepsilon_n, p^{\lambda_n}) = v_1(p^{\lambda_n} - \varepsilon).$$

But from $p^{\lambda_n} - \varepsilon > \bar{p}$ we get $v_1(p^{\lambda_n} - \varepsilon) > 0$ (because $v_1(p) > 0$ when $p > \bar{p}$). This contradicts Equation 13, since this equation implies, for $d = p^{\lambda_n} - \varepsilon$, that $v_1(p^{\lambda_n} - \varepsilon) = v^{\lambda_n}(p^{\lambda_n} - \varepsilon_n, p^{\lambda_n}) \leq 0$.

Now, assume $p < \bar{p}$. By definition, there exists a sequence of symmetric equilibria $(p^{\lambda_n}, ..., p^{\lambda_n})$ of $G^{\lambda_n}_{sym}$, which converges to $(p, ..., p)$ and which satisfies Equation 12. Taking $d = p^{max}$ in this Equation, we get

$$v^{\lambda_n}(p^{max}, p^{\lambda_n}) = 0 \leq v^{\lambda_n}(p^{\lambda_n}, p^{\lambda_n}). \tag{14}$$

But by definition

$$v^{\lambda_n}(p^{\lambda_n}, p^{\lambda_n}) = \inf_{p' \in X} \left\{ v(p^{\lambda_n}, p') + \frac{c(p', p^{\lambda_n})}{\lambda_n} \right\}.$$

Taking $p' \to (p^{\lambda_n})^+$, we get in particular

$$v^{\lambda_n}(p^{\lambda_n}, p^{\lambda_n}) \leq v_1(p^{\lambda_n}) < 0 \tag{15}$$

for n large enough (because $v_1(p) < 0$ when $p < \bar{p}$). This contradicts Equation 14.

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